

Strongly degenerate homogeneous pseudo-Kähler structures of linear type and complex plane waves

M. Castrillón López

ICMAT (CSIC-UAM-UC3M-UCM)

Departamento de Geometría y Topología

Facultad de Matemáticas, Universidad Complutense de Madrid

28040 Madrid, Spain

Ignacio Luján

Departamento de Geometría y Topología

Facultad de Matemáticas, Universidad Complutense de Madrid

28040 Madrid, Spain

Abstract

We study the class $\mathcal{K}_2 + \mathcal{K}_4$ of homogeneous pseudo-Kähler structures in the strongly degenerate case. The local form and the holonomy of a pseudo-Kähler manifold admitting such a structure is obtained, leading to a possible complex generalization of homogeneous plane waves. The same question is tackled in the case of pseudo-hyper-Kähler and pseudo-quaternion Kähler manifolds.

1 Introduction

Undoubtedly, homogeneous manifolds constitute a distinguished class of spaces on which the study of (pseudo)-Riemannian geometry is especially rich and varied. They enjoy a privileged position in Differential Geometry and have been intensively studied by means of a fruitful collection of approaches and tools. Among them, homogeneous structure tensors have proved to be one of the most successful. These tensors combine their algebraic structure together with a set of geometric PDEs known as Ambrose-Singer equations (see [2], [27]). In addition, homogeneous spaces play an essential role in different contexts of theoretical Physics as in Field Theories (cf. [11], [13]), Gravitation (cf. [12], [25]), etc.

From a geometrical point of view, homogeneous structures have been able to characterize certain spaces. For definite metrics, purely Riemannian or with additional geometry (Kähler or quaternion Kähler for in-

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stance), the so-called homogeneous structures of linear type characterize negative constant sectional (holomorphic sectional, quaternionic sectional) curvature (see [9], [17] and [27] for these results as well as indications of other similar results). When the case of metrics with signature is analyzed, the causal nature of the vector fields characterizing a homogeneous structure of linear type gives rise to different scenarios with some physical implications. In [23] and [24] the purely pseudo-Riemannian case with an isotropic structure is studied in full detail: More precisely, it is proved that these spaces have the underlying geometry of a real singular homogeneous plane wave. The aim of this paper is to extend this result to the pseudo-Kähler, pseudo-hyper-Kähler and pseudo-quaternion Kähler settings.

The main characterization of this work gives the geometry of pseudo-Kähler manifolds with a so called strongly degenerate homogeneous pseudo-Kähler structure of linear type. In particular, the expression of the metric obtained in the characterization has strong similarities with singular scale-invariant homogeneous plane waves. Furthermore, the manifolds under study and these homogeneous plane waves share some other analogies and features as it is shown in §5. Because of this, since there is not a formal definition of “complex plane wave” (as far as the authors know), pseudo-Kähler manifolds with strongly degenerate linear homogeneous structures seem to be the correct generalization of this special kind of homogeneous plane waves in complex framework, at least in the important particular Kähler case. In addition, the same techniques are successfully applied to a comparison of Cahen-Wallach spaces and one of the possible pseudo-Kähler symmetric spaces of index 2 in the classification given in [19], giving a more general picture of complex plane waves.

A relevant fact about these spaces is that they have holonomy group contained in $SU(p, q)$. Consequently, it is natural to study strongly degenerate homogeneous pseudo-hyper-Kähler and pseudo-quaternion Kähler structures of linear type. However, we prove that a manifold admitting any of those structures is necessarily flat, pointing out that the notion of homogeneous plane wave can not be realized in the pseudo-hyper-kähler or pseudo-quaternion Kähler cases in a non-trivial way.

The paper is organized as follows. In Section 2 we recall some results concerning homogeneous pseudo-Riemannian structures and give some definitions. In Section 3 the main result of the article gives the local form of the metric of a pseudo-Kähler manifold admitting a strongly degenerate homogeneous pseudo-Kähler structure of linear type. The holonomy algebra and some geometric properties are given. In addition, we analyze \mathbb{C}^{n+2} with the metric above as the local model of these manifolds. In particular, the geodesic completeness is studied showing the existence of cosmological singularities. In Section 4 we study the homogeneous model G/H associated to these kind of homogeneous structures. Geodesic completeness is again analyzed. In Section 5 we restrict ourselves to the Lorentz-Kähler case (signature $(2, 2 + 2n)$). We exhibit the relationship between pseudo-Kähler manifolds with a strongly degenerate structure of linear type and one of the possible pseudo-Kähler symmetric spaces of index 2 on one hand, and singular scale-invariant homogeneous plane waves and Cahen Wallach spaces on the other. In Section 6 the pseudo-hyper-Kähler and

pseudo-quaternion Kähler cases are studied.

2 Preliminaries

Definition 2.1 A pseudo-Riemannian manifold (M, g) is called homogeneous if there is a connected Lie group of isometries G acting transitively on M . In this case, (M, g) is called reductive homogeneous if the Lie algebra \mathfrak{g} of G can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where \mathfrak{h} is the isotropy algebra of a point $p \in M$, and $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$.

In [18], Ambrose-Singer Theorem [2] is extended to the pseudo-Riemannian setting:

Theorem 2.2 Let (M, g) be a connected, simply-connected and (geodesically) complete pseudo-Riemannian manifold. It is equivalent:

1. (M, g) is reductive homogeneous.
2. (M, g) admits a linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad (1)$$

where $S = \nabla - \tilde{\nabla}$, ∇ is the Levi-Civita connection of g , and R is the curvature tensor field of g .

A $(1, 2)$ -tensor field S satisfying equations (1) (called Ambrose-Singer equations) is called a *homogeneous pseudo-Riemannian structure*. We will also denote by S the associated $(0, 3)$ -tensor field obtained by lowering the contravariant index, $S_{XYZ} = g(S_X Y, Z)$. Let \mathcal{S} be the space of homogeneous pseudo-Riemannian structures, it is decomposed in three primitive classes

$$\begin{aligned} \mathcal{S}_1 &= \left\{ S \in \mathcal{S} : S_{XYZ} = g(X, Y)\theta(Z) - g(X, Z)\theta(Y), \theta \in \Gamma(T^*M) \right\}, \\ \mathcal{S}_2 &= \left\{ S \in \mathcal{S} : \bigotimes_{XYZ} S_{XYZ} = 0, c_{12}(S) = 0 \right\}, \\ \mathcal{S}_3 &= \left\{ S \in \mathcal{S} : S_{XYZ} + S_{YXZ} = 0 \right\}, \end{aligned}$$

where $c_{12}(S)(Z) = \sum_{i=1}^m \epsilon_i S_{e_i e_i Z}$ for any orthonormal basis $\{e_1, \dots, e_m\}$ with $g(e_i, e_i) = \epsilon_i$. A tensor field in the class \mathcal{S}_1 is called of linear type, and in addition, it is called degenerate if the vector field $\theta^\#$ is isotropic.

Let (M, g) be a pseudo-Riemannian manifold of dimension $2n$, and J a pseudo-Kähler structure, that is, a parallel pseudo-Hermitian structure with respect to the Levi-Civita connection.

Definition 2.3 A pseudo-Kähler manifold (M, g, J) is called a homogeneous pseudo-Kähler manifold if there is a connected Lie group of isometries G acting transitively on M and preserving J . In this case (M, g, J) is called reductive homogeneous pseudo-Kähler if the Lie algebra \mathfrak{g} of G can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where \mathfrak{h} is the isotropy algebra of a point $p \in M$, and $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$.

As a corollary of Kiričenko's theorem [21] we have.

Theorem 2.4 *Let (M, g, J) be a connected, simply-connected and (geodesically) complete pseudo-Kähler manifold. It is equivalent:*

1. (M, g, J) is reductive homogeneous pseudo-Kähler.
2. (M, g, J) admits a linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}J = 0, \quad (2)$$

where $S = \nabla - \tilde{\nabla}$, ∇ is the Levi-Civita connection of g , and R is the curvature tensor field of g .

A $(1, 2)$ tensor field S satisfying equations (2) is called a *homogeneous pseudo-Kähler structure*.

In [3], a classification of homogeneous pseudo-Kähler structures is obtained. Let \mathcal{K} be the space of homogeneous pseudo-Kähler structures, it is decomposed in four primitive classes

$$\begin{aligned} \mathcal{K}_1 &= \left\{ S \in \mathcal{K} : S_{XYZ} = \frac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \right. \\ &\quad \left. c_{12}(S) = 0 \right\}, \\ \mathcal{K}_2 &= \left\{ S \in \mathcal{K} : S_{XYZ} = g(X, Y)\theta_1(Z) - g(X, Z)\theta_1(Y) + g(X, JY)\theta_1(JZ) \right. \\ &\quad \left. - g(X, JZ)\theta_1(JY) - 2g(JY, Z)\theta_1(JX), \theta_1 \in \Gamma(T^*M) \right\}, \\ \mathcal{K}_3 &= \left\{ S \in \mathcal{K} : S_{XYZ} = -\frac{1}{2}(S_{YZX} + S_{ZXY} + S_{JYJZX} + S_{JZXJY}), \right. \\ &\quad \left. c_{12}(S) = 0 \right\}, \\ \mathcal{K}_4 &= \left\{ S \in \mathcal{K} : S_{XYZ} = g(X, Y)\theta_2(Z) - g(X, Z)\theta_2(Y) + g(X, JY)\theta_2(JZ) \right. \\ &\quad \left. - g(X, JZ)\theta_2(JY) + 2g(JY, Z)\theta_2(JX), \theta_2 \in \Gamma(T^*M) \right\}, \end{aligned}$$

where $c_{12}(S)(Z) = \sum_{i=1}^{2n} \epsilon_i S_{e_i e_i Z}$ for any orthonormal basis $\{e_1, \dots, e_{2n}\}$ with $g(e_i, e_i) = \epsilon_i$.

A homogeneous pseudo-Kähler structure on a pseudo-Kähler manifold (M, g, J) is called of *linear type* if it belongs to the class $\mathcal{K}_2 + \mathcal{K}_4$. The expression of a tensor field S in this class is

$$S_X Y = g(X, Y)\xi - g(Y, \xi)X - g(X, JY)J\xi + g(JY, \xi)JX - 2g(JX, \zeta)JY,$$

for some vector fields $\xi, \zeta \in \mathfrak{X}(M)$, and equations (2) are equivalent to

$$\tilde{\nabla}\xi = \tilde{\nabla}\zeta = 0, \quad \tilde{\nabla}R = 0.$$

With respect to the last condition, a simple computation shows that $\tilde{\nabla}R$ is independent of ζ so that $\xi = 0$ implies $\nabla R = 0$, that is, M is locally symmetric. Along this article, we confine ourselves to the case $\xi \neq 0$. For non-definite metrics, we thus may distinguish the following cases:

Definition 2.5 A homogeneous pseudo-Kähler structure of linear type is called:

1. non-degenerate if $g(\xi, \xi) \neq 0$.
2. weakly degenerate if $\xi \neq 0$ is isotropic and $\zeta \in \text{span}\{\xi, J\xi\}^\perp$.
3. degenerate if $\xi \neq 0$ and ζ are isotropic and $\zeta \in \text{span}\{\xi, J\xi\}^\perp$.
4. strongly degenerate if $\xi \neq 0$ is isotropic and $\zeta = 0$.

3 Strongly degenerate homogeneous structures of linear type

We confine ourselves to the strongly degenerate case, that is

$$S_X Y = g(X, Y)\xi - g(Y, \xi)X - g(X, JY)J\xi + g(JY, \xi)JX. \quad (3)$$

In [3] the following is proved.

Lemma 3.1 Let (M, g, J) be a connected pseudo-Kähler manifold of dimension $2n + 4$, $n \geq 0$, equipped with a strongly degenerate homogeneous pseudo-Kähler structure S of linear type. Let θ be the 1-form given by $\theta(X) = g(X, \xi)$. Then

1. $\nabla\theta = \theta \otimes \theta - (\theta \circ J) \otimes (\theta \circ J)$.
2. $\mathfrak{S}_{XYZ}\theta(X)R_{YZWU} = 0$ for all $X, Y, Z, W, U \in \mathfrak{X}(M)$.
3. $\nabla R = 4\theta \otimes R$, i.e., the manifold is recurrent (and hence harmonic).

Note that the anti-symmetrization of the first equation gives that $d\theta = 0$, so θ is closed. Also note that the second equation can be written as

$$\theta \wedge R(\cdot, \cdot, W, U) = 0. \quad (4)$$

Changing X, Y, Z by JX, JY, JZ we will have that

$$\mathfrak{S}_{JXJYJZ} \theta(JX)R_{JYJZWU} = 0,$$

but since R is the curvature of a pseudo-Kähler metric, this equation can be written as

$$(\theta \circ J) \wedge R(\cdot, \cdot, W, U) = 0. \quad (5)$$

We consider the complex form

$$\alpha = \theta - i(\theta \circ J),$$

which is of type $(1, 0)$ with respect to J . By direct calculation one has that $\nabla\alpha = \alpha \otimes \alpha$, so again by anti-symmetrization one obtains $d\alpha = 0$, and in particular $\bar{\partial}\alpha = 0$. Then, fixing a point $p \in M$, there exists a neighborhood U around p and a function $v : U \rightarrow \mathbb{C}$ such that $dv = \alpha$. Since α is of type $(1, 0)$ and $\alpha = dv = \partial v + \bar{\partial}v$, it must be $\bar{\partial}v = 0$, so v is holomorphic. Let $w : U \rightarrow \mathbb{C}$, $w = e^{-v}$, then

$$\nabla dw = \nabla(-e^{-v}dv) = -d(e^{-v}) \otimes dv - e^{-v} \nabla dv = e^{-v} \alpha \otimes \alpha - e^{-v} \alpha \otimes \alpha = 0. \quad (6)$$

We write $v = v^1 + iv^2$ and $w = w^1 + iw^2$ so that $w_1 = e^{-v^1} \cos v^2$ and $w^2 = -e^{-v^1} \sin v^2$, and as $dv^1 = \theta$ and $dv^2 = -\theta \circ J$ we have

$$\begin{cases} dw^1 = -(e^{-v^1} \cos v^2)\theta + (e^{-v^1} \sin v^2)(\theta \circ J) \\ dw^2 = (e^{-v^1} \sin v^2)\theta + (e^{-v^1} \cos v^2)(\theta \circ J). \end{cases} \quad (7)$$

From (4), (5) and the last two equations we have

$$\begin{cases} dw^1 \wedge R(\cdot, \cdot, W, U) = 0 \\ dw^2 \wedge R(\cdot, \cdot, W, U) = 0 \end{cases} \quad W, U \in \mathfrak{X}(M). \quad (8)$$

Let now e_{w_1}, e_{w_2} be vector fields such that $dw^i(e_{w_j}) = \delta_j^i$, from (8) we have for $i = 1, 2$

$$0 = i_{e_{w_i}}(dw^i \wedge R(\cdot, \cdot, W, U)) = R(\cdot, \cdot, W, U) - dw^i \wedge R(e_{w_i}, \cdot, W, U), \quad (9)$$

that is $R(\cdot, \cdot, W, U) = dw^i \wedge R(e_{w_i}, \cdot, W, U)$. Applying this we have

$$\begin{aligned} R(e_{w_i}, \cdot, W, U) &= R(W, U, e_{w_i}, \cdot) \\ &= W(w^i)R(e_{w_i}, U, e_{w_i}, \cdot) - U(w^i)R(e_{w_i}, W, e_{w_i}, \cdot) \end{aligned}$$

and substituting in (9) we obtain

$$\begin{aligned} R(X, Y, W, U) &= X(w^i)W(w^i)R(e_{w_i}, U, e_{w_i}, Y) \\ &\quad - X(w^i)U(w^i)R(e_{w_i}, W, e_{w_i}, Y) \\ &\quad - Y(w^i)W(w^i)R(e_{w_i}, U, e_{w_i}, X) \\ &\quad + Y(w^i)U(w^i)R(e_{w_i}, W, e_{w_i}, X), \end{aligned}$$

i.e.,

$$R = (dw^i \otimes dw^i) \wedge R(e_{w_i}, \cdot, e_{w_i}, \cdot).$$

Since this is valid for $i = 1, 2$, if we take a basis $\{e_{w_1}, e_{w_2}, e_a\}_{a=1, \dots, 2n+2}$ of $T_q M$, $q \in U$, such that $dw^i(e_a) = 0$, it follows that all the components of R with respect to this basis other than $R(e_{w_1}, e_{w_2}, e_{w_1}, e_{w_2})$ vanish.

3.1 The local form of the metric

Theorem 3.2 *Let (M, g, J) be a pseudo-Kähler manifold of dimension $2n+4$, $n \geq 0$, admitting a strongly degenerate homogeneous pseudo-Kähler structure of linear type S . Then each $p \in M$ has a neighborhood holomorphically isometric to an open subset of \mathbb{C}^{n+2} with the Kähler metric*

$$\begin{aligned} g &= dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) + \sum_{a=1}^n r_a(dx^a dw^1 + dy^a dw^2) \\ &\quad + \sum_{a=1}^n s_a(dx^a dw^2 - dy^a dw^1) + \sum_{a=1}^n (dx^a dx^a + dy^a dy^a), \end{aligned} \quad (10)$$

where $\epsilon_a = \pm 1$, and the functions b, r_a, s_a only depend on the coordinates $\{w^1, w^2\}$ and satisfy

$$\frac{\partial s_a}{\partial w^1} = \frac{\partial r_a}{\partial w^2}, \quad \frac{\partial s_a}{\partial w^2} = -\frac{\partial r_a}{\partial w^1}, \quad \Delta b = \frac{b_0}{((w^1)^2 + (w^2)^2)^2}, \quad (11)$$

for $b_0 \in \mathbb{R}$ and $a = 1, \dots, n$.

Proof. Let $p \in M$ and $w = w^1 + iw^2 : U \rightarrow \mathbb{C}$ be the holomorphic function obtained above. Since dw is not zero at some point and $\nabla dw = 0$ we have that dw is nowhere zero. Then for every $\lambda \in \mathbb{C}$, if the pre-image $\mathcal{H}_\lambda = w^{-1}(\lambda)$ is nonempty it defines a regular complex hypersurface whose tangent space is given by the kernel of dw . This means that we have our neighborhood U foliated by complex hypersurfaces \mathcal{H}_λ for λ in some open set of \mathbb{C} (note that if $w(p) = \lambda_0$ then the set $\{\lambda \in \mathbb{C} / w^{-1}(\lambda) \neq \emptyset\}$ is a neighborhood of λ_0 not containing $0 \in \mathbb{C}$).

Let $Z, JZ \in \mathfrak{X}(U)$ be such that $dw^1 = g(\cdot, Z)$ and $dw^2 = g(\cdot, JZ)$ respectively. By the expression of dw^1 and dw^2 in terms of θ and $\theta \circ J$ it is easy to see that both Z and JZ are linear combinations of ξ and $J\xi$, so they are isotropic. In addition as Z and JZ are orthogonal we have $dw(Z) = dw(JZ) = 0$. This means that Z and JZ are always tangent to the foliation given by the hypersurfaces \mathcal{H}_λ . On the other hand, from (6) we have

$$\nabla Z = \nabla JZ = 0.$$

In particular, if \mathcal{L} denotes the Lie derivative, we have

$$\begin{aligned} (\mathcal{L}_Z g)(X, Y) &= Zg(X, Y) - g([Z, X], Y) - g(X, [Y, Z]) \\ &= Zg(X, Y) - g(\nabla_Z X, Y) + g(\nabla_X Z, Y) \\ &\quad - g(X, \nabla_Z Y) + g(X, \nabla_Y Z) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} (\mathcal{L}_Z J)X &= [Z, JX] - J[Z, X] \\ &= \nabla_Z JX - \nabla_{JX} Z - J\nabla_Z X + J\nabla_X Z \\ &= 0, \end{aligned}$$

and the same for JZ . This means that Z and JZ are Killing holomorphic vector fields. Moreover, $[Z, JZ] = \nabla_Z JZ - \nabla_{JZ} Z = 0$. Reducing the neighborhood U if necessarily, we can thus take complex coordinates $\{w, z, \tilde{z}^a\}_{a=1, \dots, n}$ such that w is the function $w = w^1 + iw^2$, $Z = \partial_{z^1}$ and $JZ = \partial_{z^2}$ where $z = z^1 + iz^2$, and $\{z, \tilde{z}^a\}$ are coordinates adapted to the foliation given by \mathcal{H}_λ .

Let $\tilde{z}^a = \tilde{x}^a + i\tilde{y}^a$, it is obvious by definition that $g(\partial_{w^i}, \partial_{z^j}) = \delta_j^i$. Also since $\partial_{\tilde{x}^a}$ and $\partial_{\tilde{y}^a}$ are tangent to the foliation \mathcal{H}_λ , we have $g(\partial_{z^i}, \partial_{\tilde{x}^a}) = g(\partial_{z^i}, \partial_{\tilde{y}^a}) = 0$. Hence the metric in the real coordinates $\{w^1, w^2, z^1, z^2, \tilde{x}^a, \tilde{y}^a\}$ is given by

$$\begin{aligned} g &= dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) \\ &+ \sum_{a=1}^n \tilde{r}_a(d\tilde{x}^a dw^1 + d\tilde{y}^a dw^2) + \sum_{a=1}^n \tilde{s}_a(d\tilde{x}^a dw^2 - d\tilde{y}^a dw^1) + q(w^1, w^2), \end{aligned}$$

where $q(w^1, w^2)$ is an Hermitian metric in the coordinates $\{\tilde{x}^a, \tilde{y}^a\}$, and the functions $b, \tilde{r}_a, \tilde{s}_a$ do not depend on the coordinates z^1, z^2 as Z and JZ are Killing vector fields.

Among all the possible choices of the coordinates $\{\tilde{x}^a, \tilde{y}^a\}$ along the leaves of the foliation \mathcal{H}_λ we now construct a convenient set to simplify

the term $q(w^1, w^2)$ above. Recall that all the components of the curvature (as a $(0, 4)$ -tensor) other than $R(\partial_{w^1}, \partial_{w^2}, \partial_{w^1}, \partial_{w^2})$ vanish. With respect to the curvature $(1, 3)$ -tensor field we obtain

$$R = R(\partial_{w^1}, \partial_{w^2}, \partial_{w^1}, \partial_{w^2}) \{ (dw^1 \wedge dw^2) \otimes (dw^1 \otimes \partial_{z^2}) - (dw^1 \wedge dw^2) \otimes (dw^2 \otimes \partial_{z^1}) \}.$$

This means that the endomorphism $R_{XY} : T_p M \rightarrow T_p M$ is zero if $X, Y \notin \text{span}\{\partial_{w^1}, \partial_{w^2}\}$ and

$$\begin{aligned} R_{\partial_{w^1} \partial_{w^2}} : T_p M &\rightarrow T_p M \\ \partial_{w^1} &\mapsto R(\partial_{w^1}, \partial_{w^2}, \partial_{w^1}, \partial_{w^2}) \partial_{z^2} \\ \partial_{w^2} &\mapsto -R(\partial_{w^1}, \partial_{w^2}, \partial_{w^1}, \partial_{w^2}) \partial_{z^1} \\ W &\mapsto 0 \quad \text{if } W \notin \text{span}\{\partial_{w^1}, \partial_{w^2}\}. \end{aligned} \quad (12)$$

It is a well-known result (cf. [22, Ch.3, §9]) that the holonomy algebra \mathfrak{hol}_p of ∇ is spanned by all the elements

$$\tau^{-1} \circ R_{\tau X \tau Y} \circ \tau$$

where $X, Y \in T_p M$ and τ is the parallel displacement along an arbitrary piecewise differentiable curve starting at p . We restrict ourselves to the holonomy of the open set (U, g) . Let $E \subset T_p M$ be the subspace spanned by $\{\partial_{w^1}, \partial_{w^2}, \partial_{z^1}, \partial_{z^2}\}$ at p . Since the image of the curvature operator is contained in $\text{span}\{\partial_{z^1}, \partial_{z^2}\}$ and these are invariant by parallel displacement, we have that E is invariant by the action of the holonomy algebra $\mathfrak{hol}_p(U, g)$. Taking U simply-connected, this implies that E is invariant by the holonomy group $Hol_p(U, g)$. Since E^\perp is also invariant by the action of the holonomy and the image of the curvature operator is contained in E , the action on E^\perp is necessarily trivial. Then, let $\{(e_1)_p, (Je_1)_p, \dots, (e_n)_p, (Je_n)_p\}$ be an orthonormal basis of E^\perp , we can extend the vectors $(e_a)_p$ and $(Je_a)_p$ along U by parallel transport independently of the chosen path. In this way we obtain parallel vector fields $e_1, Je_1, \dots, e_n, Je_n$ on U . Note that since ∇ is torsionless the condition $\nabla e_a = 0 = \nabla Je_a$ implies that all these vector fields commute between them and with ∂_{z^1} and ∂_{z^2} . Besides, it implies that $\mathcal{L}_{e_a} g = 0 = \mathcal{L}_{Je_a} g$ and $\mathcal{L}_{e_a} J = 0 = \mathcal{L}_{Je_a} J$, so they are Killing holomorphic vector fields.

Let X be one of the vector fields $e_a, Je_a, \partial_{z^1}, \partial_{z^2}$, and let γ be any path starting at p . We have that

$$\frac{d}{dt}(dw(X)_{\gamma(t)}) = (\nabla_{\dot{\gamma}(t)} dw)(X_{\gamma(t)}) + dw(\nabla_{\dot{\gamma}(t)} X) = 0,$$

so the function $dw(X)$ is constant in U , and since $dw(X)_p = 0$, it is identically zero. This means that the vector fields $e_a, Je_a, \partial_{z^1}, \partial_{z^2}$ are tangent to the hypersurfaces \mathcal{H}_λ so their flows starting at a point in a hypersurface \mathcal{H}_λ remain in \mathcal{H}_λ .

This means that we can take complex coordinates $\{w, z, z^a\}$, $a = 1, \dots, n$, such that $\{z, z^a\}$, $a = 1, \dots, n$, are adapted to the foliation, and $\partial_{x^a} = e_a$, $\partial_{y^a} = Je_a$, where $z^a = x^a + iy^a$. By construction, the

metric g in the real coordinates $\{w^1, w^2, z^1, z^2, x^a, y^a\}$ is written

$$g = dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) + \sum_{a=1}^n r_a(dx^a dw^1 + dy^a dw^2) \\ + \sum_{a=1}^n s_a(dx^a dw^2 - dy^a dw^1) + \sum_{a=1}^n \epsilon_a(dx^a dx^a + dy^a dy^a), \quad (13)$$

for some functions b, r_a, s_a only depending on the variables w^1 and w^2 , and $\epsilon_a = \pm 1$. The inverse of this metric is

$$(g^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & B & 0 & -\epsilon_1 r_1 & \epsilon_1 s_1 & \dots & -\epsilon_n r_n & \epsilon_n s_n \\ 0 & 1 & 0 & B & -\epsilon_1 s_1 & -\epsilon_1 r_1 & \dots & -\epsilon_n s_n & -\epsilon_n r_n \\ 0 & 0 & -\epsilon_1 r_1 & -\epsilon_1 s_1 & \epsilon_1 & 0 & & & \\ 0 & 0 & \epsilon_1 s_1 & -\epsilon_1 r_1 & 0 & \epsilon_1 & & & \\ \vdots & \vdots & \vdots & \vdots & & & \ddots & & \\ 0 & 0 & -\epsilon_n r_n & -\epsilon_n s_n & & & & \epsilon_n & 0 \\ 0 & 0 & \epsilon_n s_n & -\epsilon_n r_n & & & & 0 & \epsilon_n \end{pmatrix} \quad (14)$$

where $B = -b + \sum_a \epsilon_a(r_a^2 + s_a^2)$. Forcing this metric to fulfill $\nabla \partial_{x^a} = \nabla \partial_{y^a} = 0$, $a = 1, \dots, n$, the following Christoffel symbols are zero

$$\Gamma_{w^1 x^a}^{z^2} = \frac{1}{2} \left\{ \frac{\partial s_a}{\partial w^1} - \frac{\partial r_a}{\partial w^2} \right\}, \quad \Gamma_{w^2 x^a}^{z^1} = \frac{1}{2} \left\{ \frac{\partial r_a}{\partial w^2} - \frac{\partial s_a}{\partial w^1} \right\}, \\ \Gamma_{w^1 y^a}^{z^2} = \frac{1}{2} \left\{ -\frac{\partial s_a}{\partial w^2} - \frac{\partial r_a}{\partial w^1} \right\}, \quad \Gamma_{w^2 y^a}^{z^1} = \frac{1}{2} \left\{ \frac{\partial r_a}{\partial w^1} + \frac{\partial s_a}{\partial w^2} \right\}.$$

Hence the functions r_a, s_a satisfy

$$\left. \begin{aligned} \frac{\partial s_a}{\partial w^1} &= \frac{\partial r_a}{\partial w^2} \\ \frac{\partial s_a}{\partial w^2} &= -\frac{\partial r_a}{\partial w^1} \end{aligned} \right\} \quad (15)$$

We can now write the Hermitian metric g in the complex coordinates $\{w, z, z^a\}$ as

$$h = \frac{1}{2} (dw \otimes d\bar{z} + dz \otimes d\bar{w}) + \frac{b}{2} dw \otimes d\bar{w} + \frac{1}{2} \sum_{a=1}^n \epsilon_a dz^a \otimes d\bar{z}^a \\ + \sum_{a=1}^n \frac{h_a}{2} dw \otimes d\bar{z}^a + \sum_{a=1}^n \frac{\bar{h}_a}{2} dz^a \otimes d\bar{w},$$

where $h_a = r_a + i(-s_a)$ is holomorphic (and \bar{h}_a is anti-holomorphic) for $a = 1, \dots, n$.

The remaining (possibly) non-zero Christoffel symbols are

$$\begin{aligned}
\Gamma_{w^1 w^2}^{z^1} &= \frac{1}{2} \frac{\partial b}{\partial w^2} - \frac{1}{2} \sum_a r_a \left(\frac{\partial r_a}{\partial w^2} + \frac{\partial s_a}{\partial w^1} \right) - \frac{1}{2} \sum_a s_a \left(-\frac{\partial s_a}{\partial w^2} + \frac{\partial r_a}{\partial w^1} \right) \\
\Gamma_{w^1 w^2}^{z^2} &= \frac{1}{2} \frac{\partial b}{\partial w^1} + \frac{1}{2} \sum_a s_a \left(\frac{\partial r_a}{\partial w^2} + \frac{\partial s_a}{\partial w^1} \right) - \frac{1}{2} \sum_a r_a \left(-\frac{\partial s_a}{\partial w^2} + \frac{\partial r_a}{\partial w^1} \right) \\
\Gamma_{w^1 w^1}^{z^1} &= \frac{1}{2} \frac{\partial b}{\partial w^1} + \sum_a (-r_a) \frac{\partial r_a}{\partial w^1} + \sum_a s_a \frac{\partial s_a}{\partial w^1} \\
\Gamma_{w^1 w^1}^{z^2} &= -\frac{1}{2} \frac{\partial b}{\partial w^2} + \sum_a s_a \frac{\partial r_a}{\partial w^1} + \sum_a r_a \frac{\partial s_a}{\partial w^1} \\
\Gamma_{w^2 w^2}^{z^1} &= -\frac{1}{2} \frac{\partial b}{\partial w^1} + \sum_a (-r_a) \frac{\partial s_a}{\partial w^2} + \sum_a (-s_a) \frac{\partial r_a}{\partial w^2} \\
\Gamma_{w^2 w^2}^{z^2} &= \frac{1}{2} \frac{\partial b}{\partial w^2} + \sum_a s_a \frac{\partial s_a}{\partial w^2} + \sum_a (-r_a) \frac{\partial r_a}{\partial w^2} \\
\Gamma_{w^1 w^2}^{x^a} &= \frac{1}{2} \left(\frac{\partial r_a}{\partial w^2} + \frac{\partial s_a}{\partial w^1} \right) \\
\Gamma_{w^1 w^1}^{x^a} &= \frac{\partial r_a}{\partial w^1} \\
\Gamma_{w^2 w^2}^{x^a} &= \frac{\partial s_a}{\partial w^2} \\
\Gamma_{w^1 w^2}^{y^a} &= \frac{1}{2} \left(-\frac{\partial s_a}{\partial w^2} + \frac{\partial r_a}{\partial w^1} \right) \\
\Gamma_{w^1 w^1}^{y^a} &= -\frac{\partial s_a}{\partial w^1} \\
\Gamma_{w^2 w^2}^{y^a} &= \frac{\partial r_a}{\partial w^2}.
\end{aligned}$$

From them, we can thus compute

$$\begin{aligned}
R(\partial_{w^1}, \partial_{w^2}, \partial_{w^1}, \partial_{w^2}) &= \frac{1}{2} \frac{\partial^2 b}{\partial w^1 \partial w^1} + \sum_a \left(\frac{1}{2} \frac{\partial s_a}{\partial w^1} \frac{\partial r_a}{\partial w^2} + \frac{1}{2} s_a \frac{\partial^2 r_a}{\partial w^1 \partial w^2} \right. \\
&+ \frac{1}{2} \left(\frac{\partial s_a}{\partial w^1} \right)^2 + \frac{1}{2} s_a \frac{\partial^2 s_a}{\partial w^1 \partial w^1} + \frac{1}{2} \frac{\partial r_a}{\partial w^1} \frac{\partial s_a}{\partial w^2} \\
&+ \frac{1}{2} r_a \frac{\partial^2 s_a}{\partial w^1 \partial w^2} - \frac{1}{2} \left(\frac{\partial r_a}{\partial w^1} \right)^2 - \frac{1}{2} r_a \frac{\partial^2 r_a}{\partial w^1 \partial w^1} \Big) \\
&+ \frac{1}{2} \frac{\partial^2 b}{\partial w^2 \partial w^2} + \sum_a \left(-\frac{\partial s_a}{\partial w^2} \frac{\partial r_a}{\partial w^1} - s_a \frac{\partial^2 r_a}{\partial w^1 \partial w^2} \right. \\
&- \frac{\partial r_a}{\partial w^2} \frac{\partial s_a}{\partial w^1} - r_a \frac{\partial^2 r_a}{\partial w^1 \partial w^2} \Big) \\
&+ \sum_a s_a \left(\frac{1}{2} \frac{\partial^2 r_a}{\partial w^1 \partial w^2} + \frac{1}{2} \frac{\partial^2 s_a}{\partial w^1 \partial w^1} - \frac{\partial^2 r_a}{\partial w^1 \partial w^2} \right) \\
&+ \sum_a r_a \left(-\frac{1}{2} \frac{\partial^2 s_a}{\partial w^1 \partial w^2} + \frac{1}{2} \frac{\partial^2 r_a}{\partial w^1 \partial w^1} + \frac{\partial^2 s_a}{\partial w^1 \partial w^2} \right).
\end{aligned}$$

Using (15) we obtain

$$R(\partial_{w^1}, \partial_{w^2}, \partial_{w^1}, \partial_{w^2}) = \frac{1}{2} \Delta b,$$

where Δ is the Laplace operator with respect to the variables (w^1, w^2) , so the curvature is

$$R = \frac{1}{2} \Delta b (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2). \quad (16)$$

One can see that all Christoffel symbols of the form $\Gamma_{..}^{w^k}$, $k = 1, 2$, vanish, whence the covariant derivative of R is

$$\begin{aligned} \nabla R &= \frac{1}{2} \frac{\partial}{\partial w^1} (\Delta b) dw^1 \otimes (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2) \\ &+ \frac{1}{2} \frac{\partial}{\partial w^2} (\Delta b) dw^2 \otimes (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2). \end{aligned} \quad (17)$$

On the other hand, solving for θ in (7) gives

$$\theta = -\frac{1}{(w^1)^2 + (w^2)^2} (w^1 dw^1 + w^2 dw^2). \quad (18)$$

Taking into account the relation (16), (17) and (18) in the equation $\nabla R = 4\theta \otimes R$ given in Lemma 3.1, we have the system of partial differential equations

$$\begin{aligned} \frac{\partial}{\partial w^1} (\Delta b) &= \frac{-4w^1}{(w^1)^2 + (w^2)^2} \Delta b, \\ \frac{\partial}{\partial w^2} (\Delta b) &= \frac{-4w^2}{(w^1)^2 + (w^2)^2} \Delta b, \end{aligned}$$

which can be integrated to give the Poisson equation

$$\Delta b = \frac{b_0}{((w^1)^2 + (w^2)^2)^2},$$

for some constant $b_0 \in \mathbb{R}$. ■

Corollary 3.3 *The curvature of the metric g with respect to the coordinates $\{w^1, w^2, z^1, z^2, x^a, y^a\}$ is*

$$R = \frac{1}{2} \frac{b_0}{((w^1)^2 + (w^2)^2)^2} (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2) \quad b_0 \in \mathbb{R}, \quad (19)$$

hence (M, g, J) is a pseudo-Kähler Ricci-flat manifold.

Note that (M, g, J) is flat if and only if $b_0 = 0$.

Corollary 3.4 *In complex coordinates $w = w^1 + iw^2$, $z = z^1 + iz^2$, $z^a = x^a + iy^a$, $a = 1, \dots, n$, the metric (10) is expressed*

$$\begin{aligned} h &= \frac{1}{2} (dw \otimes d\bar{z} + dz \otimes d\bar{w}) + \frac{b}{2} dw \otimes d\bar{w} + \sum_{a=1}^n \frac{h_a}{2} dw \otimes d\bar{z}^a \\ &+ \sum_{a=1}^n \frac{\bar{h}_a}{2} dz^a \otimes d\bar{w} + \frac{1}{2} \sum_{a=1}^n \epsilon_a dz^a \otimes d\bar{z}^a, \end{aligned} \quad (20)$$

where $\epsilon_a = \pm 1$, $h_a : \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function of the variable w for all $a = 1, \dots, n$, and $b : \mathbb{C} \rightarrow \mathbb{R}$ is a function of w satisfying the Poisson equation

$$\Delta b = \frac{b_0}{||w||^4}$$

for some $b_0 \in \mathbb{R}$.

3.2 Some global properties

In this section we derive some special properties of a pseudo-Kähler manifold (M, g, J) admitting a strongly degenerate homogeneous pseudo-Kähler structure of linear type:

Proposition 3.5 *If $b_0 \neq 0$, the holonomy algebra of (M, g, J) can be identified with the one dimensional space*

$$\mathfrak{hol} = \mathbb{R} \begin{pmatrix} i & i & 0 \\ -i & -i & 0 \\ 0 & 0 & 0_n \end{pmatrix} \subset \mathfrak{su}(1, 1) \subset \mathfrak{su}(p, q).$$

Proof. The fact that the manifold is Ricci-flat implies that the restricted holonomy is contained in $SU(p, q)$ if the signature of (M, g) is $(2p, 2q)$. Moreover, in the proof of Theorem 3.2 it is shown that the holonomy group of the neighborhood (U, g) at p has two invariant subspaces: $E = \text{span}\{\partial_{w^1}, \partial_{w^2}, \partial_{z^1}, \partial_{z^1}\}$ and E^\perp . This means that the local holonomy representation is decomposable into the trivial representation on E^\perp and a representation of a group $H \subset SU(1, 1)$ on E . On the other hand, since (M, g) is a complex manifold, it is in particular real analytic, so the restricted holonomy, the local holonomy and the infinitesimal holonomy coincides (see [22, Vol.I, Ch. II]). Recall that the infinitesimal holonomy algebra at $p \in M$ is defined as $\mathfrak{hol}' = \bigcup_{k=0}^\infty \mathfrak{m}_k$, where

$$\mathfrak{m}_0 = \text{span}\{R_{XY}/X, Y \in T_p M\}$$

and

$$\mathfrak{m}_k = \text{span}\{\mathfrak{m}_{k-1} \cup \{(\nabla_{Z_k} \dots \nabla_{Z_1} R)_{XY}/Z_1, \dots, Z_k, X, Y \in T_p M\}\}.$$

In our case, by the recurrent formula $\nabla R = 4\theta \otimes R$ we have $\mathfrak{m}_k = \mathfrak{m}_{k-1} = \dots = \mathfrak{m}_1 = \mathfrak{m}_0$, hence $\mathfrak{hol}' = \mathfrak{m}_0$. In addition, by (12) \mathfrak{m}_0 is the one dimensional space

$$\mathfrak{m}_0 = \text{span}\{R_{\partial_{w^1} \partial_{w^2}}\} = \text{span}\{A\}$$

where

$$\begin{aligned} A : \quad T_p M &\rightarrow T_p M \\ \partial_{w^1} &\mapsto \partial_{z^2} \\ \partial_{w^2} &\mapsto -\partial_{z^1} \\ \partial_{z^1}, \partial_{z^2} &\mapsto 0 \\ \partial_{x^a}, \partial_{y^a} &\mapsto 0. \end{aligned} \tag{21}$$

Let s be the sign of b at p . With respect to the unitary basis $\{W, Z\}$ of E defined as

$$W = \frac{\partial_{w^1}}{\sqrt{|b|}} - i \frac{\partial_{w^2}}{\sqrt{|b|}}, \quad Z = \left(\frac{\partial_{w^1}}{\sqrt{|b|}} - s\sqrt{|b|}\partial_{z^1} \right) - i \left(\frac{\partial_{w^2}}{\sqrt{|b|}} - s\sqrt{|b|}\partial_{z^2} \right),$$

the matrix of the endomorphism A restricted to E is

$$A|_E = s \begin{pmatrix} i & i \\ -i & -i \end{pmatrix} \in \mathfrak{su}(1, 1).$$

■

Proposition 3.6 (M, g, J) is a Walker manifold.

Proof. Recall that the distribution of 2-planes spanned by ∂_{z^1} and ∂_{z^2} is invariant by holonomy. Since ξ and $J\xi$ are linear combination of ∂_{z^1} and ∂_{z^2} , these vector fields generates a null parallel distribution. Therefore, (M, g, J) is a Walker manifold (see [7]). ■

A *scalar invariant* is a scalar function obtained by fully contracting the curvature R and its derivatives $\nabla^k R$ with the metric and the inverse of the metric.

Proposition 3.7 (M, g, J) is VSI (vanishing scalar invariants).

Proof. Since the curvature (19) only involves dw^1 and dw^2 , it is easy to see that the inverse metric (14) forces all possible scalar invariants to vanish. ■

Proposition 3.8 (M, g, J) is an Osserman manifold with a 2-step nilpotent Jacobi operator.

Proof. The Jacobi operators, that is $J(X) : Y \mapsto R_{YX}X$, of the elements of the basis $\{\partial_{w^1}, \partial_{w^2}, \partial_{z^1}, \partial_{z^2}, \partial_{x^a}, \partial_{y^a}\}$ are

$$\begin{aligned} J(\partial_{w^1}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{b_0}{2||w||^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{2n} \end{pmatrix}, \\ J(\partial_{w^2}) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{b_0}{2||w||^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{2n} \end{pmatrix}, \\ J(\partial_{z^1}) &= 0, \quad J(\partial_{z^2}) = 0, \quad J(\partial_{x^a}) = 0, \quad J(\partial_{y^a}) = 0. \end{aligned}$$

■

3.3 The manifold (\mathbb{C}^{2+n}, h)

Theorem 3.2 gives the local expression (24) of the metric of a manifold with a strongly degenerate homogeneous Kähler structure of linear type. This motivates the study of the space \mathbb{C}^{2+n} endowed with this particular pseudo-Kähler metric, which can thus be understood as the simplest instance of this type of manifolds. In particular, the goal of this section is to

study the singular nature of this space, and to explicitly exhibit the corresponding strongly degenerate homogeneous pseudo-Kähler structures of linear type. In addition, this results will be applied in §5.

We thus consider (\mathbb{C}^{2+n}, h) with the standard complex structure of \mathbb{C}^{2+n} and the pseudo-Kähler metric h given in (20). With respect to the standard real coordinates $\{w^1, w^2, z^1, z^2, x^a, y^a\}$, $a = 1, \dots, n$, of $\mathbb{C}^{2+n} \equiv \mathbb{R}^{4+2n}$, the metric has the form (13) where the functions b, r_a, s_a only depend on the variables w^1 and w^2 and satisfy

$$\left. \begin{aligned} \frac{\partial s_a}{\partial w^1} &= \frac{\partial r_a}{\partial w^2} \\ \frac{\partial s_a}{\partial w^2} &= -\frac{\partial r_a}{\partial w^1} \\ \Delta b &= \frac{b_0}{((w^1)^2 + (w^2)^2)^2}, \quad b_0 \in \mathbb{R}. \end{aligned} \right\} \quad (22)$$

Note, that the $(0,3)$ -Riemann curvature tensor is

$$R = \frac{1}{2} \frac{b_0}{((w^1)^2 + (w^2)^2)^2} \left((dw^1 \wedge dw^2) \otimes (dw^1 \otimes \partial_{z^2}) - (dw^1 \wedge dw^2) \otimes (dw^2 \otimes \partial_{z^1}) \right),$$

so if $b_0 \neq 0$ it exhibits a singular behavior at $(w^1, w^2) = (0, 0)$. The set $\{w^1 = w^2 = 0\}$ can be understood as a singularity of g in the cosmological sense:

1. The geodesic deviation equation is governed by the components of the curvature tensor $R_{w^1 w^2 w^i}^{z^j}$, $i, j = 1, 2$, making the tidal forces infinite at $\{w^1 = w^2 = 0\}$.
2. The geodesic equation for the variables (w^1, w^2) are

$$\ddot{w}^1 = 0, \quad \ddot{w}^2 = 0.$$

Then, geodesics with initial value

$$(w^1(0), w^2(0)) = (a, b), \quad (\dot{w}^1(0), \dot{w}^2(0)) = (c, cb/a),$$

with $b \in \mathbb{R}$, $a, c \in \mathbb{R} - \{0\}$, are

$$w^1(t) = ct + a, \quad w^2(t) = c \frac{b}{a} t + b,$$

and geodesics with initial value

$$(w^1(0), w^2(0)) = (0, b), \quad (\dot{w}^1(0), \dot{w}^2(0)) = (0, d),$$

with $b, d \in \mathbb{R} - \{0\}$, are

$$w^1(t) = 0, \quad w^2(t) = dt + b.$$

These geodesics reach the singular set $\{w^1 = w^2 = 0\}$ in finite time $t = -\frac{a}{c}$ and $t = -\frac{b}{d}$ respectively. Hence $(\mathbb{C}^{2+n} - \{w^1 = w^2 = 0\}, h)$ is not geodesically complete.

There exists the possibility that this singularity is due to a bad choice of coordinates so that we could embed $(\mathbb{C}^{2+n} - \{w^1 = w^2 = 0\}, h)$ in a complete manifold (\hat{M}, \hat{h}) with \hat{h} of class \mathcal{C}^2 . To see that this is actually not possible, we can compute a component of the curvature tensor with respect to an orthonormal parallel frame along a curve reaching the singular set in finite time, and see that it is singular (see [26]). Indeed, let γ be the geodesic with initial value $\gamma(0) = (1, 0, \dots, 0)$ and $\dot{\gamma} = (-1, 0, \dots, 0)$. We have seen that this geodesic is of the form

$$\gamma(t) = (1 - t, 0, z^1(t), z^2(t), x^a(t), y^a(t))$$

for some functions $z^1(t), z^2(t), x^a(t), y^a(t)$, $a = 1, \dots, n$. Let

$$E(t) = W^1(t)\partial_{w^1} + W^2(t)\partial_{w^2} + Z^1(t)\partial_{z^1} + Z^2(t)\partial_{z^2} + X^a(t)\partial_{x^a} + Y^a(t)\partial_{y^a}$$

be a vector field along γ . E is parallel, i.e. $\nabla_{\dot{\gamma}} E = 0$, if the following equations hold:

$$\begin{aligned} 0 &= \dot{W}^1, & 0 &= \dot{W}^2, \\ 0 &= \dot{Z}^1 - W^1\Gamma_{w^1 w^1}^{z^1} - W^2\Gamma_{w^1 w^2}^{z^1}, & 0 &= \dot{Z}^2 - W^1\Gamma_{w^1 w^1}^{z^2} - W^2\Gamma_{w^1 w^2}^{z^2}, \\ 0 &= \dot{X}^a - W^1\Gamma_{w^1 w^1}^{x^a} - W^2\Gamma_{w^1 w^2}^{x^a}, & 0 &= \dot{Y}^a - W^1\Gamma_{w^1 w^1}^{y^a} - W^2\Gamma_{w^1 w^2}^{y^a}. \end{aligned}$$

We can thus obtain an orthonormal parallel frame $\{E_1(t), \dots, E_{4+2n}(t)\}$ with $E_1(t)$ and $E_2(t)$ of the form

$$\begin{aligned} E_1(t) &= \frac{1}{\sqrt{|b(0)|}}\partial_{w^1} + Z_1^1(t)\partial_{z^1} + Z_1^2(t)\partial_{z^2} + X_1^a(t)\partial_{x^a} + Y_1^a(t)\partial_{y^a}, \\ E_2(t) &= \frac{1}{\sqrt{|b(0)|}}\partial_{w^2} + Z_2^1(t)\partial_{z^1} + Z_2^2(t)\partial_{z^2} + X_2^a(t)\partial_{x^a} + Y_2^a(t)\partial_{y^a}, \end{aligned}$$

where $E_1(0) = \frac{1}{\sqrt{|b(0)|}}\partial_{w^1}$, $E_2(0) = \frac{1}{\sqrt{|b(0)|}}\partial_{w^2}$, and $b(0) = b(1, 0, \dots, 0)$.

The value of the curvature tensor applied to $E_1(t), E_2(t)$ is

$$R_{E_1(t)E_2(t)E_1(t)E_2(t)} = \frac{b_0}{2b(0)^2} \frac{1}{(w^1(t)^2 + w^2(t)^2)^2} = \frac{b_0}{2b(0)^2} \frac{1}{(1-t)^4},$$

which is singular at $t = 1$, that is when γ reaches the singular set $\{w^1 = w^2 = 0\}$.

Finally we show that strongly degenerate homogeneous pseudo-Kähler structures of linear type indeed exist and are realized in the manifold $(\mathbb{C}^{2+n} - \{w^1 = w^2 = 0\}, h)$.

Proposition 3.9 *For every data (b, b_0, r_a, s_a) , $a = 1, \dots, n$, satisfying (22), the pseudo-Kähler manifold $(\mathbb{C}^{2+n} - \{w^1 = w^2 = 0\}, h)$ admits a strongly degenerate pseudo-Kähler homogeneous structure of linear type.*

Proof. Let S be the tensor field

$$S_X Y = g(X, Y)\xi - g(Y, \xi)X - g(X, JY)J\xi + g(JY, \xi)JX,$$

with

$$\xi = \frac{-1}{(w^1)^2 + (w^2)^2}(w^1\partial_{z^1} + w^2\partial_{z^2}).$$

It is a straightforward computation to see that $\tilde{\nabla}\xi = 0$ and $\tilde{\nabla}R = 0$, where $\tilde{\nabla} = \nabla - S$, so that S satisfies equations (2). ■

4 The homogeneous model for a strongly degenerate homogeneous structure of linear type

Let (M, g, J) be a reductive homogeneous pseudo-Kähler manifold admitting a strongly degenerate homogeneous structure of linear type S . From [27] one can construct a Lie algebra of infinitesimal isometries associated to S . This algebra is (fixing a point $p \in M$ as the origin)

$$\mathfrak{g} = T_p M \oplus \mathfrak{hol}^{\tilde{\nabla}}$$

where $\tilde{\nabla} = \nabla - S$ is the canonical connection associated to the homogeneous structure tensor S . The brackets in \mathfrak{g} are

$$\begin{cases} [A, B] &= AB - BA, & A, B \in \mathfrak{hol}^{\tilde{\nabla}} \\ [A, \eta] &= A \cdot \eta, & A \in \mathfrak{hol}^{\tilde{\nabla}}, \eta \in T_p M \\ [\eta, \zeta] &= S_\eta \zeta - S_\zeta \eta + \tilde{R}_{\eta\zeta}, & \eta, \zeta \in T_p M, \end{cases}$$

where \tilde{R} is the curvature tensor of $\tilde{\nabla}$. This curvature tensor can be computed as $\tilde{R} = R - R^S$ where

$$R_{XY}^S Z = [S_X, S_Y] Z - S_{S_X Y - S_Y X} Z.$$

Let G be a Lie group with Lie algebra \mathfrak{g} , and let H be the connected Lie subgroup with Lie algebra $\mathfrak{hol}^{\tilde{\nabla}}$. If H is closed in G , then G/H is called a *homogeneous model* for M , which means that (M, g, J) is locally holomorphically isometric to G/H with the G -invariant metric and complex structure given by g and J at $T_p M$.

By direct calculation from (3) one finds

$$R_{XY}^S Z = -2g(X, JY) (g(\xi, JZ)\xi + g(Z, \xi)J\xi),$$

which is readily seen to equal

$$\left(R_p^S\right)_{XY} = -2g(X, JY)A,$$

where A is the endomorphism given in (21). In terms of the coordinate system $\{w^1, w^2, z^1, z^2, x^a, y^a\}$ around p described in Theorem 3.2 (with p sent to $(1, 0, \dots, 0) \in \mathbb{C}^{n+2}$), and taking into account (12), we can write

$$\tilde{R}_p = \left(\frac{b_0}{2} dw^1 \wedge dw^2 + 2\omega_p\right) \otimes A,$$

where ω is the Kähler form associated to (g, J) . This means that $\mathfrak{hol}^{\tilde{\nabla}} = \mathfrak{hol} \simeq \mathfrak{u}(1) \subset \mathfrak{su}(1, 1)$.

The (real) Lie algebra \mathfrak{g} is then

$$\mathfrak{g} \simeq \mathfrak{u}(1) \oplus \mathbb{C}^2 \oplus \mathbb{C}^n,$$

where $\mathbb{C}^2 = \mathbb{C}\{\partial_{w^1}, \partial_{z^1}\}$ and $\mathbb{C}^n = \mathbb{C}\{\partial_{x^a}, a = 1, \dots, n\}$. Setting $B = \partial_{z^2} - A$, the brackets are

$$\begin{aligned}
[A_1, A_2] &= 0, \\
[B, \partial_{z^1}] &= 0, \quad [B, \partial_{z^2}] = 0, \quad [B, \partial_{x^a}] = 0, \quad [B, \partial_{y^a}] = 0, \\
[B, \partial_{w^1}] &= 2B, \quad [B, \partial_{w^2}] = 0, \\
[\partial_{z^1}, \partial_{z^2}] &= 0, \quad [\partial_{w^1}, \partial_{w^2}] = (-2b(p) + \frac{1}{2}b_0)B - \frac{1}{2}b_0\partial_{z^2} \\
[\partial_{z^1}, \partial_{w^1}] &= \partial_{z^1}, \quad [\partial_{z^1}, \partial_{w^2}] = \partial_{z^2} - 2B, \\
[\partial_{z^2}, \partial_{w^1}] &= \partial_{z^2} + 2B, \quad [\partial_{z^2}, \partial_{w^2}] = -\partial_{z^1}, \\
[\partial_{x^a}, \partial_{x^b}] &= [\partial_{y^a}, \partial_{y^b}] = 0, \\
[\partial_{x^a}, \partial_{y^b}] &= -2\delta_b^a B, \\
[\partial_{z^k}, \partial_{x^a}] &= [\partial_{z^k}, \partial_{y^a}] = 0, \quad k = 1, 2, \\
[\partial_{w^1}, \partial_{x^a}] &= -\partial_{x^a}, \quad [\partial_{w^1}, \partial_{y^a}] = -\partial_{y^a}, \\
[\partial_{w^2}, \partial_{x^a}] &= -\partial_{y^a}, \quad [\partial_{w^2}, \partial_{y^a}] = \partial_{x^a},
\end{aligned} \tag{23}$$

for $a, b = 1, \dots, n$ and $A_1, A_2 \in \mathfrak{u}(1)$.

One can check that \mathfrak{g} is a solvable Lie algebra with a 2-step nilradical $\mathfrak{n} = \mathbb{R}\{A, \partial_{z^1}, \partial_{z^2}, \partial_{x^a}, \partial_{y^a}, a = 1, \dots, n\}$. Note also that \mathfrak{g} contains a Heisenberg algebra $\mathfrak{h} = \mathbb{R}\{B, \partial_{x^a}, \partial_{y^a}, a = 1, \dots, n\}$.

Using Lie's Theorem ([16]) we can obtain an upper triangular representation of the Lie algebra \mathfrak{g} as shown in Table 1, where $t, w_1, w_2, z_1, z_2, x_1, y_1, \dots, x_n, y_n \in \mathbb{R}$, $\lambda = 2b(p)$ and $\mu = \frac{b_0}{2}$.

The isotropy algebra is generated by the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0
\end{pmatrix},$$

so exponentiating we obtain a Lie group G and a closed subgroup H defining a homogeneous model for M . The metric g and the complex structure J give a bilinear form and a complex structure on

$$\mathfrak{m} = \text{span}\{\partial_{w^1}, \partial_{w^2}, \partial_{z^1}, \partial_{z^2}, \partial_{x^a}, \partial_{y^a}\}$$

respectively. These induce a G -invariant metric \bar{g} and a G -invariant complex structure \bar{J} on G/H so that (M, g, J) is locally holomorphically isometric to $(G/H, \bar{g}, \bar{J})$.

Proposition 4.1 *The homogeneous model $(G/H, \bar{g}, \bar{J})$ constructed above is not geodesically complete.*

Proof. Let σ be the Lie algebra involution of \mathfrak{g} given by

$$\begin{aligned}
\sigma : \quad \mathfrak{g} &\rightarrow \mathfrak{g} \\
B &\mapsto -B \\
\partial_{w^1} &\mapsto \partial_{w^1} \\
\partial_{w^2} &\mapsto -\partial_{w^2} \\
\partial_{z^1} &\mapsto \partial_{z^1} \\
\partial_{z^2} &\mapsto -\partial_{z^2} \\
\partial_{x^a} &\mapsto \partial_{x^a} \\
\partial_{y^a} &\mapsto -\partial_{y^a}.
\end{aligned}$$

$-2w_1$	$2w_2 + 2w_1i$	$2w_2 + 2w_1i$	$-4y_1i$	$-4x_1i$	\dots	$-4y_ni$	$-4x_ni$	$2t + (\lambda - \mu)w_2 + 2(z_1 - z_2)i$	$-4 + (\mu - \lambda)w_1$
0	$-w_1 + w_2i$	0	0	0	\dots	0	0	$z_1 - \frac{\mu}{2}w_2i$	$\frac{\mu}{2}w_1i$
0	0	$-w_1 - w_2i$	0	0	\dots	0	0	$z_2 - \frac{\mu}{2}w_2i$	$z_1 + z_2i - \frac{\mu}{2}w_1i$
0	0	0	$-w_1 + w_2i$	0	\dots	0	0	x_1	$-x_1i$
0	0	0	0	$-w_1 + w_2i$	\dots	0	0	y_1	y_1i
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots	\vdots
0	0	0	0	0	\dots	$-w_1 + w_2i$	0	x_n	$-x_ni$
0	0	0	0	0	\dots	0	$-w_1 + w_2i$	y_n	y_ni
0	0	0	0	0	\dots	0	0	0	0
0	0	0	0	0	\dots	0	0	0	0

Table 1: Lie algebra \mathfrak{g} of G

One can check that the restriction of σ to \mathfrak{m} is an isometry with respect to the bilinear form given by \bar{g} . The subalgebra of fixed points is $\mathfrak{g}^\sigma = \mathbb{R}\{\partial_{w^1}, \partial_{z^1}, \partial_{x^a}\}$. Working with the universal cover if necessary we can assume that G is simply-connected so that σ induces an involution in G and therefore an isometric involution in G/H . We will denote all this involutions by σ . Let G^σ be the connected Lie subgroup of G with Lie algebra \mathfrak{g}^σ , note that $\mathfrak{g}^\sigma \cap \mathfrak{hol}^\nabla = \{0\}$, so $(G/H)^\sigma = G^\sigma$, where $(G/H)^\sigma$ stands for the fixed point set of $\sigma : G/H \rightarrow G/H$. It is a well-known result that $(G/H)^\sigma = G^\sigma$ is a closed totally geodesic submanifold of G/H . Let now θ be the Lie algebra involution of \mathfrak{g}^σ given by

$$\begin{aligned} \theta : \quad \mathfrak{g}^\sigma &\rightarrow \mathfrak{g}^\sigma \\ \partial_{w^1} &\mapsto \partial_{w^1} \\ \partial_{z^1} &\mapsto \partial_{z^1} \\ \partial_{x^a} &\mapsto -\partial_{x^a}, \end{aligned}$$

which is again an isometry with respect to the bilinear form induced in \mathfrak{g}^σ by restriction from \mathfrak{m} . The subalgebra of fixed points is $\mathfrak{k} = (\mathfrak{g}^\sigma)^\theta = \mathbb{R}\{\partial_{w^1}, \partial_{z^1}\}$. Let \widetilde{G}^σ be the universal cover of G^σ , $\theta : \mathfrak{g}^\sigma \rightarrow \mathfrak{g}^\sigma$ induces an isometric involution $\theta : \widetilde{G}^\sigma \rightarrow \widetilde{G}^\sigma$. Therefore, let K be the connected Lie subgroup of \widetilde{G}^σ with lie algebra \mathfrak{k} , K is a totally geodesic submanifold of \widetilde{G}^σ .

Let s be the sign of $b(p)$. We define the left-invariant vector fields in \mathfrak{k}

$$U = \frac{1}{\sqrt{|b(p)|}} \partial_{w^1}, \quad V = U - s\sqrt{|b(p)|} \partial_{z^1}.$$

We have

$$\langle U, U \rangle = s, \quad \langle V, V \rangle = -s, \quad \langle U, V \rangle = 0,$$

$$[U, V] = \frac{1}{\sqrt{|b(p)|}}(u - v),$$

where $\langle \cdot, \cdot \rangle$ stands for the bilinear form inherited by \mathfrak{k} from \mathfrak{g}^σ and which determines the pseudo-Riemannian left-invariant metric in K . The Levi-Civita connection of this metric is

$$\begin{aligned} \nabla_U U &= \frac{1}{\sqrt{|b(p)|}} V, & \nabla_U V &= \frac{1}{\sqrt{|b(p)|}} U, \\ \nabla_V V &= \frac{1}{\sqrt{|b(p)|}} U, & \nabla_V U &= \frac{1}{\sqrt{|b(p)|}} V. \end{aligned}$$

Let γ be a curve in K and $\dot{\gamma}$ its tangent vector. Setting $\dot{\gamma}(t) = u(t)U + v(t)V$, the geodesic equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ implies

$$\begin{aligned} \dot{u} + \frac{1}{\sqrt{|b(p)|}}(uv + v^2) &= 0 \\ \dot{v} + \frac{1}{\sqrt{|b(p)|}}(uv + u^2) &= 0. \end{aligned}$$

Changing variables to $x = u + v$ and $y = u - v$ the equations transform into

$$\begin{aligned} \dot{x} + \frac{1}{\sqrt{|b(p)|}} x^2 &= 0 \\ \dot{y} - \frac{1}{\sqrt{|b(p)|}} xy &= 0, \end{aligned}$$

the solutions of which are

$$x = \sqrt{|b(p)|} \frac{1}{t-c}, \quad y = A \exp \left(\sqrt{|b(p)|} \frac{1}{t-c} \right)$$

for some constants $A, c \in \mathbb{R}$. Therefore, K is not geodesically complete. Hence, since we have the following inclusions of totally geodesic submanifolds

$$K \subset \widetilde{G}^\sigma, \quad G^\sigma = (G/H)^\sigma \subset G/H,$$

the manifold $(G/H, g, J)$ is not geodesically complete. ■

Corollary 4.2 *Let (M, g, J) be a connected and simply-connected pseudo-Kähler manifold admitting a strongly degenerate pseudo-Kähler structure of linear type S , then it is geodesically uncomplete.*

Proof. Suppose that (M, g, J) is geodesically complete. Ambrose-Singer theorem assures that (M, g, J) is (globally) holomorphically isometric to the homogeneous model $(G/H, \bar{g}, \bar{J})$. But this homogeneous model is not geodesically complete. ■

5 Homogeneous structures and homogeneous plane waves

5.1 The Lorentz case

Definition 5.1 *A plane wave is the Lorentz manifold $M = \mathbb{R}^{n+2}$ with metric*

$$g = dudv + A_{ab}(u)x^a x^b du^2 + \sum_{a=1}^n (dx^a)^2,$$

where (A_{ab}) is a symmetric matrix called the profile.

A plane wave is called *homogeneous* if the Lie algebra of Killing vector fields acts transitively in the tangent space at every point. It is well known [6] that any plane wave admits the following algebra of Killing vector fields

$$\text{span}\{\partial_v, X_{p_a}, X_{q_a}; a = 1, \dots, n\},$$

where p_a, q_a are solutions of the harmonic oscillator equation $d^2 f/du^2 = A(u)f$ with initial values

$$\begin{aligned} (p_a)_b(u_0) &= \delta_{ab}, & (\dot{p}_a)_b(u_0) &= 0 \\ (q_a)_b(u_0) &= 0, & (\dot{q}_a)_b(u_0) &= \delta_{ab}, \end{aligned}$$

and

$$X_f = f_a \partial_{x^a} - (df/du)_a x^a \partial_v.$$

In particular, this algebra is isomorphic to the Heisenberg algebra. Therefore, the homogeneity of a plane wave is characterized by the existence of an extra Killing vector field with non zero u -component. Homogeneous plane waves were classified in [6]. Among them there are two special types in which we are interested: Cahen-Wallach spaces and singular scale-invariant homogeneous plane waves.

The Cahen-Wallach space $M_{\lambda_1, \dots, \lambda_n}^{1, n+1}$ is defined as a plane wave with metric

$$g = dudv + A_{ab} x^a x^b du^2 + \sum_{a=1}^n (dx^a)^2,$$

where (A_{ab}) is a constant symmetric matrix with eigenvalues $(\lambda_1, \dots, \lambda_n)$. They are one of the possible simply connected Lorentzian symmetric spaces together with $(\mathbb{R}, -dt^2)$, the de Sitter, and the anti de Sitter spaces (see [8]). Note that the curvature information of a plane wave is contained in the profile A , this meaning that the only non-vanishing component is

$$R_{uaub} = -A_{ab}(u).$$

The condition of being symmetric is then

$$\nabla R = 0 \Leftrightarrow \partial_u A_{ab} = 0,$$

which is obviously satisfied. As a symmetric space a Cahen-Wallach space admits the homogeneous pseudo-Riemannian structure $S = 0$. The extra Killing vector field is $X = \partial_u$.

A singular scale-invariant homogeneous plane wave is a plane wave with metric

$$g = dudv + \frac{B_{ab}}{u^2} x^a x^b du^2 + \sum_{a=1}^n (dx^a)^2,$$

where (B_{ab}) is a constant symmetric matrix. Unlike Cahen-Wallach spaces these kind of plane waves are not geodesically complete. Moreover, as their name suggests, these spaces are homogeneous with extra Killing vector field $X = u\partial_u - v\partial_v$, but not symmetric since the profile $A(u) = B/u^2$ is not u -independent. They enjoy many properties. For example, they have been found to occur universally as Pentose Limits of space-time singularities [5]. Furthermore in [24] the following characterization is given.

Theorem 5.2 *Let (M, g) be a connected pseudo-Riemannian manifold of dimension $n+2$ admitting a degenerate homogeneous pseudo-Riemannian structure of liner type, i.e., $S_X Y = g(X, Y)\xi - g(\xi, Y)X$ with $g(\xi, \xi) = 0$. Then (M, g) is locally isometric to \mathbb{R}^{n+2} with metric*

$$ds^2 = dudv + \frac{B_{ab}}{u^2} x^a x^b du^2 + \sum_{a=1}^n \varepsilon_a (dx^a)^2$$

for some symmetric matrix (B_{ab}) and $\varepsilon_a = \pm 1, a = 1, \dots, n$.

Note that for Lorentzian signature this means that a manifold admitting a degenerate homogeneous structure of linear type is locally a singular scale-invariant homogeneous plane wave. Conversely it is easy to see that every singular scale-invariant homogeneous plane wave admits such a homogeneous structure with $\xi = -\frac{1}{u}\partial_v$.

5.2 The Lorentz-Kähler case

By a Lorentz-Kähler manifold we understand a pseudo-Kähler manifold of index 2. In this subsection we will exhibit the relation and similarities between Cahen-Wallach spaces and singular scale-invariant homogeneous plane waves on one side and some kind of (locally) homogeneous Lorentz-Kähler manifolds on the other. Although, as far as the authors know, there is no formal definition of a “complex” plane wave, this relation could allow us to understand the latter spaces as a complex generalization of the former, at least in the important Lorentz-Kähler case, suggesting a starting point for a possible definition of *complex plane waves*.

Cahen-Wallach spaces are the model of symmetric Lorentzian plane waves. Furthermore, as a wave, it is the twisted product of a plane wave front and a two dimensional manifold containing time and the direction of propagation. This two dimensional space gives the real geometric information of the total manifold and in particular it contains a null parallel vector field. For this reason, in the Lorentz-Kähler case, we study symmetric manifolds of complex dimension two (real dimension four) with a null parallel one complex dimensional distribution. In the classification of simply-connected indecomposable and not irreducible pseudo-Kählerian symmetric spaces of signature $(2, 2)$ given in [19] (see also [20]), there is just one possibility corresponding to such a situation. This is a manifold with holonomy algebra

$$\mathfrak{ho}_{n=0}^{\gamma_1=0, \gamma_2=0} = \mathbb{R}p_1 \wedge p_2 = \mathbb{R} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and curvature $R_{\lambda_5=1}$ or $-R_{\lambda_5=1}$ in the notation of [19] which we explain now. Here the tangent space has been identified with $\mathbb{R}^{2,2} = \mathbb{C}^{1,1}$ with a basis $\{p_1, p_2, q_1, q_2\}$ with respect to which the metric and the complex structure take form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

respectively. The curvature $R_{\lambda_5=1}$ stands for the curvature operator $R \in S^2(\mathbb{R}^{2,2} \wedge \mathbb{R}^{2,2})$ that sends all elements of the basis to zero except for $R(q_1 \wedge q_2) = p_1 \wedge p_2$.

Now, in order to get the Kähler-Lorentz analog of Cahen-Wallach spaces, we add a n complex dimensional plane wave front to this four dimensional manifold. This is done by considering the holonomy algebra $\mathfrak{ho}_{n=0}^{\gamma_1=0, \gamma_2=0} \oplus \{\text{Id}_{2n}\}$ acting in $\mathbb{R}^{2n+4} = \mathbb{R}^{2,2} \oplus \mathbb{R}^{2n}$ through the action of $\mathfrak{ho}_{n=0}^{\gamma_1=0, \gamma_2=0}$ in $\mathbb{R}^{2,2}$ and the trivial action in \mathbb{R}^{2n} .

Proposition 5.3 *Let (M, g, J) be a (locally) symmetric Lorentz-Kähler manifold of dimension $2n+4$, $n \geq 0$, with holonomy $\mathfrak{ho}_{n=0}^{\gamma_1=0, \gamma_2=0} \oplus \{\text{Id}_{2n}\}$*

acting on $T_x M \simeq \mathbb{R}^{2n+4}$, $x \in M$, as explained above. The metric g is locally of the form

$$g = dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) + \sum_{a=1}^n r_a(dx^a dw^1 + dy^a dw^2) + \sum_{a=1}^n s_a(dx^a dw^2 - dy^a dw^1) + \sum_{a=1}^n (dx^a dx^a + dy^a dy^a), \quad (24)$$

where the functions b, r_a, s_a , $a = 1, \dots, n$, only depend on w^1 and w^2 and satisfy

$$\frac{\partial s_a}{\partial w^1} = \frac{\partial r_a}{\partial w^2}, \quad \frac{\partial s_a}{\partial w^2} = -\frac{\partial r_a}{\partial w^1}, \quad \Delta b = b_0, \quad b_0 \in \mathbb{R} - \{0\}.$$

Proof. First note that since $\nabla R = 0$ the holonomy algebra at a point x is generated by the elements R_{XY} , $X, Y \in T_x M$. Let $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ be an orthonormal basis of \mathbb{R}^{2n} , since p_1, p_2, e_i, Je_i , $i = 1, \dots, n$, are invariant by holonomy, we can extend them by parallel transport defining parallel vector fields Z, JZ, E_i, JE_i . Let now $\alpha^1 = g(\cdot, Z)$ and $\alpha^2 = -\alpha^1 \circ J$, consider the complex form $\alpha = \alpha^1 + i\alpha^2$. Since $\nabla Z = 0 = \nabla JZ$, we have $\nabla \alpha = 0$, hence in particular α is holomorphic and closed. This means that locally there is a holomorphic function $w : U \rightarrow \mathbb{C}$ such that $dw = \alpha$. Since dw is non-zero at some point and it is parallel, we have that dw is never zero. Hence if the set $w^{-1}(\lambda)$, $\lambda \in \mathbb{C}$, is non-empty it defines a complex hypersurface in U . Note that since $g(Z, Z) = 0 = g(Z, E_i) = g(Z, JE_i)$ the vector fields Z, JZ, E_i, JE_i are always tangent to the hypersurfaces $w^{-1}(\lambda)$. Therefore we can take coordinates $\{w^1, w^2, z^1, z^2, x^i, y^i\}$ such that $w = w^1 + iw^2$, $\partial_{z^1} = Z$, $\partial_{z^2} = JZ$, $\partial_{x^i} = E_i$ and $\partial_{y^i} = JE_i$. With respect to this coordinates the metric is

$$g = dw^1 dz^1 + dw^2 dz^2 + b(dw^1 dw^1 + dw^2 dw^2) + \sum_{a=1}^n r_a(dx^a dw^1 + dy^a dw^2) + \sum_{a=1}^n s_a(dx^a dw^2 - dy^a dw^1) + \sum_{a=1}^n (dx^a dx^a + dy^a dy^a),$$

where the functions b, r_a, s_a only depend on w^1 and w^2 . Now imposing $\nabla \partial_{x^a} = 0 = \nabla \partial_{y^a}$ we obtain

$$\begin{aligned} \frac{\partial s_a}{\partial w^1} &= \frac{\partial r_a}{\partial w^2} \\ \frac{\partial s_a}{\partial w^2} &= -\frac{\partial r_a}{\partial w^1}. \end{aligned} \quad (25)$$

In addition, the only non-zero element of the curvature tensor is

$$R_{\partial w^1 \partial w^2 \partial w^1 \partial w^2} = \frac{1}{2} \Delta b,$$

where Δ stands for the Laplace operator with respect to the variables (w^1, w^2) . The condition of being (locally) symmetric is then

$$\nabla R = 0 \Leftrightarrow \Delta b = b_0,$$

for $b_0 \in \mathbb{R} - \{0\}$. ■

In view of this Proposition we consider the pseudo-Kähler manifold (\mathbb{C}^{2+n}, g) , with g given by (24), as a natural Lorentz-Kähler equivalent to Cahen-Wallach spaces. Note that the Laplacian condition admits solutions with singularities. As Cahen-Wallach spaces are simply-connected, we only consider solutions b defined on the whole \mathbb{C}^{2+n} . For these b , it is easy to check that (\mathbb{C}^{2+n}, g) is geodesically complete.

We now study Lorentzian singular scale-invariant homogeneous plane waves. As these manifolds are characterized by degenerate homogeneous structure tensors of linear type (see §5.1), from Theorem 3.2 above the natural equivalent to this spaces are Lorentz-Kähler manifolds with strongly degenerate homogeneous pseudo-Kähler structure tensors of linear type, and more precisely, the space $(\mathbb{C}^{n+2} - \{w^1 = w^2 = 0\}, g)$ with g given by (10). Moreover, the local expression of the metric (10) given in Theorem 3.2 (restricted to signature $(2, 2 + 2n)$) and the metric (24) are the same except for the function b , which has a different Laplacian in each case. As a straight forward computation shows, the curvature tensor of both metrics (24) and (10) is

$$R = \frac{1}{2} \Delta b (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2),$$

so all the curvature information is contained in the Laplacian of the function b . For this reason, analogously to Lorentz plane waves, we call Δb the profile of the metric. It is worth noting that in the Lorentz case one goes from Cahen-Wallach spaces to singular scale-invariant homogeneous plane waves by making the profile be singular with a term $1/u^2$. Doing so, the space is no longer geodesically complete and a cosmological singularity at $\{u = 0\}$ is created. In the same way, in the Lorentz-Kähler case one goes from metric (24) to (10) by making the profile be singular with a term $1/((w^1)^2 + (w^2)^2)^2$ and again one transforms a geodesically complete space to a geodesically uncomplete space, and a cosmological singularity at $\{w^1 = w^2 = 0\}$ is created. This exhibits a close relation between this two couples of spaces.

	Symmetric space	Strongly Deg. homog. of linear type
Lorentz	Cahen-Wallach spaces Profile: $A(u) = A(const.)$ Geodesically complete	Singular s.-i. homog. plane wave Profile: $A(u) = A/u^2$ Geodesically uncomplete
Lorentz-Kähler	\mathbb{C}^{2+n} with metric (24) Profile: $\Delta b = b_0(const.)$ Geodesically complete	$\mathbb{C}^{2+n} - \{w = 0\}$ with metric (10) Profile: $\Delta b = b_0/ w ^4$ Geodesically uncomplete

Finally, if we now write the expression of the metrics (24) and (10) in complex notation (see Corollary 20 above) we get

$$\begin{aligned} h = & \frac{1}{2} (dw \otimes d\bar{z} + dz \otimes d\bar{w}) + \frac{b}{2} dw \otimes d\bar{w} + \sum_{a=1}^n \frac{h_a}{2} dw \otimes d\bar{z}^a \\ & + \sum_{a=1}^n \frac{\bar{h}_a}{2} dz^a \otimes d\bar{w} + \frac{1}{2} \sum_{a=1}^n dz^a \otimes d\bar{z}^a. \end{aligned}$$

This expressions obey the general formula of a *pp-wave* (plane wave front with parallel rays) (see e.g. [4], [14]) but written with complex coordinates instead of real coordinates. This kind of Lorentz manifolds include plane waves and are related to (gravitational) radiation propagating at the speed of light. Plane waves are exact solutions of Einstein's field equations and metrics (24) and (24) are Ricci-flat, so they solve vacuum Einstein's field equations. Finally, it is worth noting that metrics (24) and (10) are VSI (see Proposition 3.7) a common property of all plane waves.

6 The pseudo-hyper-Kähler and pseudo-quaternion Kähler case

During this section $\dim(M) = 4n \geq 8$ is assumed. We shall study strongly degenerate homogeneous structures of linear type in the pseudo-hyper-Kähler and the pseudo-quaternion Kähler cases.

Definition 6.1 *Let (M, g) be a pseudo-Riemannian manifold. A pseudo-quaternionic Hermitian structure is a 3-rank subbundle $v^3 \subset \mathfrak{so}(TM)$ with a local basis J_1, J_2, J_3 satisfying*

$$J_1^2 = J_2^2 = J_3^2 = -1, \quad J_1 J_2 = J_3.$$

This means that at every point $p \in M$ there is a subalgebra $v_p^3 \subset \mathfrak{so}(T_p M)$ isomorphic to the imaginary quaternions, and in particular g has signature $(4p, 4q)$.

Definition 6.2 *A pseudo-Riemannian manifold (M, g) is called pseudo-quaternion Kähler if it admits a parallel pseudo-quaternionic Hermitian structure with respect to the Levi-Civita connection, or equivalently if the holonomy group of the Levi-Civita connection is contained in $Sp(p, q)Sp(1)$.*

Let J_1, J_2, J_3 be a local basis of v^3 , and $\omega_a = g(\cdot, J_a \cdot)$, $a = 1, 2, 3$. The 4-form

$$\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

is independent of the choice of basis and hence it is globally defined. A pseudo-quaternionic Hermitian manifold (M, g, v^3) is pseudo-quaternion Kähler if and only if Ω is parallel with respect to the Levi-Civita connection (cf. [1]).

Definition 6.3 *An pseudo-quaternion Kähler manifold (M, g, v^3) is called a homogeneous pseudo-quaternion Kähler manifold if there is a connected Lie group G of isometries acting transitively on M and preserving*

v^3 . (M, g, v^3) is called a *reductive homogeneous pseudo-quaternion Kähler manifold* if the Lie algebra \mathfrak{g} of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

As a corollary of Kiričenko's Theorem [21] we have

Theorem 6.4 *A connected, simply connected and (geodesically) complete pseudo-quaternion Kähler manifold (M, g, v^3) is reductive homogeneous if and only if it admits a linear connection $\tilde{\nabla}$ satisfying*

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}\Omega = 0, \quad (26)$$

where $S = \nabla - \tilde{\nabla}$, ∇ is the Levi-Civita connection, R is the curvature tensor of ∇ , and Ω is the canonical 4-form associated to v^3 .

A tensor field S satisfying the previous equations is called a *homogeneous pseudo-quaternion Kähler structure*. The classification of such structures was obtained in [3], resulting five primitive classes $\mathcal{QK}_1, \mathcal{QK}_2, \mathcal{QK}_3, \mathcal{QK}_4, \mathcal{QK}_5$. Among them $\mathcal{QK}_1, \mathcal{QK}_2, \mathcal{QK}_3$ have dimension growing linearly with respect to the dimension of M . Hence

Definition 6.5 *A homogeneous pseudo-quaternion Kähler structure S is called of linear type if it belongs to the class $\mathcal{QK}_1 + \mathcal{QK}_2 + \mathcal{QK}_3$.*

The local expression of $S \in \mathcal{QK}_1 + \mathcal{QK}_2 + \mathcal{QK}_3$ is

$$\begin{aligned} S_X Y = g(X, Y)\xi - g(Y, \xi)X + \sum_{a=1}^3 (g(J_a Y, \xi)J_a X - g(X, J_a Y)J_a \xi) \\ + \sum_{a=1}^3 g(X, \zeta^a)J_a Y, \end{aligned} \quad (27)$$

where ξ and ζ^a , $a = 1, 2, 3$, are vector fields. We then give the following further definition.

Definition 6.6 *A homogeneous pseudo-quaternion Kähler structure of linear type given by formula (27) is called strongly degenerate if $\xi \neq 0$, $g(\xi, \xi) = 0$ and $\zeta^a = 0$ for $a = 1, 2, 3$.*

Proposition 6.7 *Let (M, g, v^3) be a pseudo-quaternion Kähler manifold admitting a strongly degenerate homogeneous pseudo-quaternion Kähler structure of linear type. Then (M, g, v^3) is flat.*

Proof. Let S be a strongly degenerate homogeneous pseudo-quaternion Kähler structure of linear type on (M, g, v^3) . From (27) and the third equation in (26) we have

$$\nabla_X \xi = S_X \xi = g(X, \xi)\xi - \sum_{a=1}^n g(X, J_a \xi)J_a \xi,$$

and

$$\nabla_X J_a \xi = \tilde{\tau}^c(X)J_b \xi - \tilde{\tau}^b(X)J_c \xi + g(X, J_a \xi)\xi - \sum_{d=1}^3 g(X, J_d J_a \xi)J_d \xi$$

for certain 1-forms $\tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3$ defined by the parallel property of v^3 , where (a, b, c) is any cyclic permutation of $(1, 2, 3)$. Using this formulas and after a long calculation one obtains

$$R_{XY}\xi = \nabla_{[X,Y]}\xi - [\nabla_X, \nabla_Y]\xi = 0.$$

Since (M, g, v^3) is Einstein we have

$$0 = r(X, \xi) = \nu_q g(X, \xi), \quad X \in \mathfrak{X}(M),$$

where ν_q is one-quarter of the reduced scalar curvature. Supposing that $\xi \neq 0$ this implies that $\nu_q = 0$ and hence (M, g, v^3) is Ricci-flat. Therefore the manifold is locally hyper-Kähler and the curvature R is of type $\mathfrak{sp}(p, q)$, i.e., $R_{XYJ_aZW} + R_{XZJ_aYW} = 0$ for $a = 1, 2, 3$.

Now, the second equation in (26) reads

$$(\nabla_X R)_{YZWU} = -R_{S_XYZWU} - R_{YS_XZWU} - R_{YZS_XWU} - R_{YZWS_XU},$$

so taking the cyclic sum in X, Y, Z and applying Bianchi identities, after some computations we get

$$\begin{aligned} 0 &= \mathfrak{S}_{XYZ} \left\{ g(\xi, Y)R_{XZWU} - \sum_{a=1}^3 g(\xi, J_a Y)R_{J_a XZWU} \right. \\ &\quad + g(\xi, Z)R_{YXWU} - \sum_{a=1}^3 g(\xi, J_a Z)R_{YJ_a XWU} \\ &\quad + g(\xi, W)R_{YZXU} - \sum_{a=1}^3 g(\xi, J_a W)R_{YZJ_a XU} \\ &\quad \left. + g(\xi, U)R_{YZWX} - \sum_{a=1}^3 g(\xi, J_a U)R_{YZWJ_a X} \right\} \\ &= 2 \mathfrak{S}_{XYZ} g(X, \xi)R_{ZYWU}. \end{aligned}$$

This means that $\theta \wedge R_{WU} = 0$ with $\theta = \xi^\flat$. But since R is of type $\mathfrak{sp}(p, q)$ we also have

$$\begin{cases} (\theta \circ J_1) \wedge R_{WU} = 0 \\ (\theta \circ J_2) \wedge R_{WU} = 0 \\ (\theta \circ J_3) \wedge R_{WU} = 0. \end{cases}$$

It is easy to see that these equations force $R_{WU} = 0$, and hence (M, g, v^3) must be flat. ■

Definition 6.8 *Let (M, g) be a pseudo-Riemannian manifold. A pseudo-hyper-Hermitian structure is a subset $\{J_1, J_2, J_3\} \subset \mathfrak{so}(TM)$ satisfying*

$$J_1^2 = J_2^2 = J_3^2 = -1, \quad J_1 J_2 = J_3.$$

In particular g has signature $(4p, 4q)$.

Definition 6.9 A pseudo-Riemannian manifold (M, g) is called *pseudo-hyper-Kähler* if it admits a parallel pseudo-hyper-Hermitian structure with respect to the Levi-Civita connection, or equivalently if the holonomy group of the Levi-Civita connection is contained in $Sp(p, q)$.

Definition 6.10 A pseudo-hyper-Kähler manifold (M, g, J_1, J_2, J_3) is called a *homogeneous pseudo-hyper-Kähler manifold* if it admits a connected Lie group G of isometries acting transitively on M and preserving J_a , $a = 1, \dots, 3$. (M, g, J_1, J_2, J_3) is called a *reductive homogeneous pseudo-hyper-Kähler manifold* if the Lie algebra \mathfrak{g} of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

It is worth noting that, unlike the Riemannian setting, a homogeneous and Ricci-flat pseudo-Riemannian manifold is not necessarily flat. As a corollary of Kiričenko's Theorem [21] we have again

Theorem 6.11 A connected, simply connected and (geodesically) complete pseudo-hyper-Kähler manifold (M, g, J_a) is reductive homogeneous if and only if it admits a linear connection $\tilde{\nabla}$ satisfying

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad \tilde{\nabla}J_a = 0, \quad a = 1, \dots, 3 \quad (28)$$

where $S = \nabla - \tilde{\nabla}$, ∇ is the Levi-Civita connection and R is the curvature tensor of ∇ .

A tensor field S satisfying the previous equations is called a *homogeneous pseudo-hyper-Kähler structure*. The classification of these structures is obtained in [10], resulting three primitive classes $\mathcal{HK}_1, \mathcal{HK}_2, \mathcal{HK}_3$. Among them only \mathcal{HK}_1 has dimension growing linearly with respect to the dimension of M , hence we call a homogeneous pseudo-hyper-Kähler structure of *linear type* if it belongs to the class \mathcal{HK}_1 . The expression of these tensors is

$$S_X Y = g(X, Y)\xi - g(Y, \xi)X + \sum_{a=1}^3 (g(J_a Y, \xi)J_a X - g(X, J_a Y)J_a \xi),$$

where ξ is a vector field. Analogously to the previous cases, S is called strongly degenerate if $\xi \neq 0$ is isotropic.

Let now (M, g, J_a) be a pseudo-hyper-Kähler manifold admitting a strongly degenerate homogeneous pseudo-hyper-Kähler structure of linear type. Repeating exactly the same computations as in the pseudo-quaternion Kähler case, but knowing a priori that $\nu_q = 0$, we arrive to the conclusion that (M, g, J_a) must be flat.

Remark 6.12 Note that in the pseudo-Kähler case we have proved (see Proposition 3.5) that admitting a strongly degenerate structure of linear type automatically implies that the manifold has an integrable $SU(p, q)$ structure. Moreover $SU(p, q)$ -homogeneous structures of linear type have the same expression as strongly degenerate pseudo-Kähler structures. So the $SU(p, q)$ case is already done. Homogeneous manifolds with other geometric structures, such as $G_{2(2)}^*$ or $Spin(4, 3)$, can also admit structures of linear type, namely there is a submodule of the space of homogeneous structures with dimension growing linearly with the dimension of the manifold. The study of such structures is open.

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